LECTURE NOTES

PROGRAMME – BCA

SEMESTER- IV

DISCRETE MATHEMATICS (BCA-401)

UNIT IV

Unit-IV (10)

Propositional Logic:

Proposition, well-formed formula, Truth tables, Tautology, Satisfiability, Contradiction ,Algebra of proposition, Theory of Inference. Predicate Logic: First order predicate, well formed formula of predicate, quantifiers, Inference theory of predicate logic.

PROPOSITIONS AND LOGICAL OPERATIONS

2.0 OBJECTIVES :

After going through this unit, you will be able to :

- Define statement & logical operations.
- Define & to use the laws of Logic.
- Describe the logical equivalence and implications.
- Define arguments & valid arguments.
- Test the validity of argument using rules of logic.
- Give proof by truth tables.
- Give proof by mathematical Induction.

2.1 INTRODUCTION :

Mathematics is an exact science. Every statement in Mathematics must be precise. Also there can't be Mathematics without proofs and each proof needs proper reasoning. Proper reasoning involves logic. The dictionary meaning of 'Logic' is the science of reasoning. The rules of logic gives precise meaning to mathematic statements. These rules are used to distinguished between valid & invalid mathematical arguments.

In addition to its importance in mathematical reasoning, logic has numerous applications in computer science to verify the correctness of programs & to prove the theorems in natural & physical sciences to draw conclusion from experiments, in social sciences & in our daily lives to solve a multitude of problems.

The area of logic that deals with propositions is called the propositional calculus or propositional logic. The mathematical approach to logic was first discussed by British mathematician George Boole; hence the mathematical logic is also called as Boolean logic.

In this chapter we will discuss a few basic ideas.

2.2 PROPOSITIONS (OR STATEMENTS)

A proposition (or a statement) is a declarative sentence that is either true or false, but not both.

A proposition (or a statement) is a declarative sentence which is either true or false but not both.

Imperative, exclamatory, interrogative or open statements are not statements in logic. Mathematical identities are considered to be statements.

Example 1 : For Example consider, the following sentences.

i) The earth is round.

ii) $4 + 3 = 7$

iii) London is in Denmark

iv) Do your homework

v) Where are you going?

vi) $2 + 4 = 8$

vii) $15 < 4$

viii) The square of 4 is 18.

ix) $x + 1 = 2$

x) May God Bless you!

All of them are propositions except iv), v), ix) & x) sentences i), ii) are true, where as iii), iv), vii) & viii) are false.

Sentence iv) is command hence not proposition. Is a question so not a statement. ix) Is a declarative sentence but not a statement, since it is true or false depending on the value of x. x) is a exclamatory sentence and so it is not statement.

Mathematical identities are considered to be statements.

Statements which are imperative, exclamatory, interrogative or open are not statements in logic.

Compound statements :

Many propositions are composites that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound propositions.

A proposition is said to be primitive if it can not be broken down into simpler propositions, that is, if it is not composite.

Example 2 : Consider, for example following sentences.

(a) "The sum is shining today and it is cold"

(b) "Juilee is intelligent or studies every night."

Also the propositions in Example 1 are primitive propositions.

2.3 LOGICAL OPERATIONS OR LOGICAL CONNECTIVES :

The phrases or words which combine simple statements are called logical connectives.

For example, 'and', 'or', 'note', 'if……then', 'either…….or' etc….

In the following table we list some possible connectives, their symbols & the nature of the compound statement formed by them.

Sr. No.	Connective	Symbol	Compound statement
1	AND	$\wedge$	Conjuction
2	OR	$\vee$	Disjunction
3	NOT	$\neg$	Negation
4	If..........then	$\rightarrow$	Conditional or implication
5	If and only if (iff)	$\leftrightarrow$	Biconditional or equivalence

Now we shall study each of basic logical connectives in details.

Basic Logical Connectives :

2.3.1 Conjunction (AND) :

If two statements are combined by the word "and" to form a compound proposition (statement) is called the conjunction of the original proposition.

Symbolically, if P & Q are two simple statements, then '$P \wedge Q$' denotes the conjunction of P and Q and is read as 'P and Q.

Since, $P \wedge Q$ is a proposition it has a truth value and this truth value depends only on the truth values of P and Q.

Specifically, if P & Q are true then $P \wedge Q$ is true; otherwise $P \wedge Q$ is false.

The truth table for conjunction is as follows.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3 :

Let P : Monsoon is very good this year.
Q : The rivers are rising.

then
$P \wedge Q$: Monsoon is very good this year and rivers are rising.

2.3.2 Disjunction (OR) :

Any two statements can be connected by the word 'or' to form a compound statement called disjunction.

Symbolically, if P and Q are two simple statements, then $P \vee Q$ denotes the disjunction of P and Q and read as 'P or Q'.

The truth value of $P \vee Q$ depends only on the truth values of P and Q. specifically if P and Q are false then $P \vee Q$ is false otherwise $P \vee Q$ is true.

The truth table for disjunction is as follows.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 4 :

P : Paris is in France
Q : $2 + 3 = 6$
then $P \vee Q$: Paris is in France or 2 + 3 = 6.
Here, $P \vee Q$ is <u>True</u> since P is true & Q is False.

Thus, the disjunction $P \vee Q$ is false only when P and Q are both false.

2.3.3 Negation (NOT)

Given any proposition P, another proposition, called negation of P, can be formed by writing "It is not the case that…….. or". "It is false that……." before P or, if possible, by inserting in P the word "not".

Symbolically $\neg P$ or $\sim P$ read "not P" denotes the negation of P. the truth value of $\neg P$ depends on the truth value of P.

If P is true then $\neg P$ is false and if P is false then $\neg P$ is true. The truth table for Negation is as follows :

P	$\neg P$
T	F
F	T

Example 5 :

Let P : 6 is a factor of 12.
Then Q = $\neg P$: 4 is not a factor of 12.
Here P is true & $\neg P$ is false.

2.3.4 Conditional or Implication : (If……then)

If two statements are combined by using the logical connective 'if….then' then the resulting statement is called a conditional statement.

If P and Q are two statements forming the implication "if P then Q" then we denotes this implication $P \to Q$.

In the implication $P \to Q$,

P is called antecedent or hypothesis

Q is called consequent or conclusion.

The statement $P \to Q$ is true in all cases except when P is true and Q is false.

The truth table for implication is as follows.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Since conditional statement play an essential role in mathematical reasoning a variety of terminology is used to express $P \to Q$.

i) If P then Q

ii) P implies Q

iii) P only if Q

iv) Q if P

v) P is sufficient condition for Q

vi) Q when P

vii) Q is necessary for P

viii) Q follows from P

ix) if P, Q

x) Q unless $\neg P$

Converse, Inverse and Contrapositive of a conditional statement : We can form some new conditional statements starting with a conditional statements related conditional statements that occur so often that they have special names $\to$ converse, contrapositive & Inverse. Starting with a conditional statement $P \to Q$ that occur so often that they have special names.

1. **Converse :** If $P \to Q$ is an implication then $Q \to P$ is called the converse of $P \to Q$.

2. **Contrapositive :** If $P \to Q$ is an implication then the implication $\neg Q \to \neg P$ is called it's contrapositive.

3. **Inverse :** If $P \to Q$ is an implication then $\neg P \to \neg Q$ is called its inverse.

Example 6 :

Let P : You are good in Mathematics.
Q : You are good in Logic.

Then, $P \to Q$: If you are good in Mathematics then you are good in Logic.

1) Converse : $(Q \to P)$
   If you are good in Logic then you are good in Mathematics.

2) Contrapositive : $\neg Q \to \neg P$
   If you are not good in Logic then you are not good in Mathematics.

3) Inverse : $(\neg P \to \neg Q)$
   If you are not good in Mathematics then you are not good in Logic.

2.3.5 Biconditional Statement : Let P and Q be propositions. The biconditional statement $P \leftrightarrow Q$ is the proposition "P if and only if Q". The biconditional statement is true when P and Q have same truth values and is false otherwise.

Biconditional statements are also called bi-implications. It is also read as p is necessary and sufficient condition for Q.

The truth table for biconditional statement is as follows.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 7 : Let P : You can take the flight.
Q : You buy a ticket.
Then $P \leftrightarrow Q$ is the statement.
"You can take the flight iff you buy a ticket".

Precedence of Logical Operators :

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. In order to avoid an excessive number of parantheses.

We sometimes adopt an order of precedence for the logical connectives. The following table displays the precedence levels of the logical operators.

Operator	**Precedence**
$\neg$	1
$\wedge$	2
$\wedge$	3
$\rightarrow$	4
$\leftrightarrow$	5

2.4 LOGICAL EQUIVALANCE :

Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Definition : The compound propositions P and Q are called logically equivalent if $P \leftrightarrow Q$ is a tautology. The notation $P \equiv Q$ denotes that P and Q are logically equivalent.

Some equivalance are useful for deducing other equivalance. The following table shows some important equivalance.

2.4.1 Logical Identities or Laws of Logic :

Name	**Equivalance**
1. Identity Laws	$P \wedge T \equiv P$
	$P \vee F \equiv P$
2. Domination Laws	$P \vee T \equiv T$
	$P \wedge F \equiv F$
3. Double Negation	$\neg(\neg P) \equiv P$
4. Idempotent Laws	$P \vee P \equiv P$
	$P \wedge P \equiv P$
5. Commutative Laws	$P \vee Q \equiv Q \vee P$
	$P \wedge Q \equiv Q \wedge P$
6. Associative Laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

7. Distributive Laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
8. De Morgan's Laws	$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$
9. Absorption Laws	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$
10. Negation Laws (Inverse / Complement)	$P \vee \neg P \equiv T$ $P \wedge \neg P \equiv F$
11. Equivalance Law	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$
12. Implication Law	$P \to Q \equiv \neg P \vee Q$
13. Biconditional Property	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$
14. Contrapositive of Conditional statement	$P \to Q \equiv \neg Q \to \neg P$

Note that while taking negation of compound statement 'every' or 'All' is interchanged by 'some' & 'there exists' is interchanged by 'at least one' & vice versa.

Example 8 : If P : "This book is good."

Q : "This book is costly."

Write the following statements in symbolic form.

a) This book is good & costly.
b) This book is not good but costly.
c) This book is cheap but good.
d) This book is neither good nor costly.
e) If this book is good then it is costly.

Answers :

a) $P \wedge Q$
b) $\neg P \wedge Q$
c) $\neg Q \wedge P$
d) $\neg P \wedge \neg Q$
e) $P \to Q$

2.4.2 Functionally complete set of Connectives :

We know that there are five logical connectives $\neg, \vee, \wedge, \to$ and $\leftrightarrow$. But some of these can be expressed in terms of the other & we get a smaller set of connectives.

The set containing minimum number of connectives which are sufficient to express any logical formula in symbolic form is called as the functionally complete set of connectives.

There are following two functionally complete set of connectives.

(1) $\{\neg, \vee\}$ is complete set connectives.

Here, the $\wedge$ can be expressed using $\neg$ & $\vee$.

$\therefore P \wedge Q \equiv \neg\ \neg(P \wedge Q)$

$\equiv \neg\ (\neg P \vee\ \neg Q)$

The $\rightarrow$ can be expressed in terms of $\neg, \vee$.

$\therefore P \rightarrow Q \equiv \neg P \vee Q$

The $\leftrightarrow$ can be expressed in terms of $\neg, \vee$

$\therefore\ P \leftrightarrow Q \quad \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$

$\equiv \neg\left[\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)\right]$

$\therefore \{\neg, \vee\}$ is a functionally complete set of connectives.

Similarly, you can prove that $\therefore \{\neg, \wedge\}$ is complete set of connectives.

2.5 LOGICAL IMPLICATIONS:

A proposition P (p, q, ……..) is said to logically imply a proposition Q (p, q, …….) written,

P (p, q, ……..) $\Rightarrow$ Q (p, q, …….) if Q (p, q, …….) is true whenever P (p, q, …….) is true.

Example 9 : $P \Rightarrow (P \vee Q)$

Solution :

Consider the truth table for this

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Observe that if P is true (T) in rows 1 and 2 then $P \vee Q$ is also true (T).
$\therefore P \Rightarrow P \vee Q$.

If Q (p, q, …….) is true whenever P (p, q, …….) is true then the argument. $P(p, q, ...) \vdash Q(p, q, ...)$ is valid and conversely.

i.e. the argument $P \vdash Q$ is valid iff the conditional statement $P \rightarrow Q$ is always true, i.e. a tautology.

2.5.1 Logical Equivalence Involving Implications :

Let P & Q be two statements.

The following table displays some useful equivalences for implications involving conditional and biconditional statements.

Sr. No.	Logical Equivalence involving implications
1	$P \rightarrow Q \equiv \neg P \vee Q$
2	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
3	$P \vee Q \equiv \neg P \rightarrow Q$
4	$P \wedge Q \equiv \neg(P \rightarrow \neg Q)$
5	$\neg(P \rightarrow Q) \equiv P \wedge \neg Q$
6	$(P \rightarrow Q) \wedge (P \rightarrow r) \equiv P \rightarrow (Q \wedge r)$
7	$(P \rightarrow r) \wedge (Q \rightarrow r) \equiv (P \vee Q) \rightarrow r$
8	$(P \rightarrow Q) \vee (P \rightarrow r) \equiv P \rightarrow (Q \vee r)$
9	$(P \rightarrow r) \vee (Q \rightarrow r) \equiv (P \wedge Q) r$
10	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
11	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$
12	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$
13	$\neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$

All these identities can be proved by using truth tables.

2.6 NORMAL FORM AND TRUTH TABLES :

2.6.1 Well Formed Formulas : (wff)

A compound statement obtained from statement letters by using one or more connectives is called a statement pattern or statement form. thus, if P, Q, R, ……. are the statements (which can be treated as variables) then any statement involving these statements and the logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ is a statement form or a well formed formula or statement pattern.

Definition : A propositional variable is a symbol representing any proposition. Note that a propositional variable is not a proposition but can be replaced by a proposition.

Any statement involving a propositional variable and logical connectives is a well formed formula.

Note : A wff is not a proposition but we substitute the proposition in place of propositional variable, we get a proposition.

E.g. $\neg(P \vee Q) \wedge (\neg Q \wedge R) \rightarrow Q, \neg(P \rightarrow Q)$ etc.

2.6.1 (a) Truth table for a Well Formed Formula :

If we replace the propositional variables in a formula α by propositions, we get a proposition involving connectives. If α involves n propositional constants, we get 2n possible combination of truth variables of proposition replacing the variables.

Example 10 : Obtain truth value for $\alpha = (P \rightarrow Q) \wedge (Q \rightarrow P)$.

Solution : The truth table for the given well formed formula is given below.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	α
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

2.6.1 (b) Tautology :

A tautology or universally true formula is a well formed formula, whose truth value is T for all possible assignments of truth values to the propositional variables.

Example 11 : Consider $P \vee \neg P$, the truth table is as follows.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

$\therefore P \vee \neg P$ always takes value T for all possible truth value of P, it is a tautology.

2.6.1 (c) Contradiction :

A contradiction or (absurdity) is a well formed formula whose truth value is false (F) for all possible assignments of truth values to the propositional variables.

Thus, in short a compound statement that is always false is a contradiction.

Example 12 : Consider the truth table for $P \wedge \neg P$.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

$\therefore P \wedge \neg P$ always takes value F for all possible truth values of P, it is a contradiction.

2.6.1. (d) Contingency :

A well formed formula which is neither a tautology nor a contradiction is called a contingency.

Thus, contingency is a statement pattern which is either true or false depending on the truth values of its component statement.

Example 13 : Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution : The truth tables for these compound proposition is as follows.

1	2	3	4	5	6	7	8
P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$6 \leftrightarrow 7$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

We cab observe that the truth values of $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q.

It follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology, therefore the given compound propositions are logically equivalent.

Example 14 : Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution : The truth tables for these compound proposition as follows.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

As the truth values of $p \to q$ and $\neg p \vee q$ are logically equivalent.

Example 15 : Determine whether each of the following form is a tautology or a contradiction or neither :

i) $(P \wedge Q) \to (P \vee Q)$

ii) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$

iii) $(\neg P \wedge \neg Q) \to (P \to Q)$

iv) $(P \to Q) \wedge (P \wedge \neg Q)$

v) $\lceil \lfloor P \wedge (P \to \neg Q) \to Q \rceil \rfloor$

Solution:

i) The truth table for $(p \wedge q) \to (p \vee q)$

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

 All the entries in the last column are ‘T’.

$\therefore (p \wedge q) \to (p \vee q)$ is a tautology.

ii) The truth table for $(p \vee q) \wedge (\neg p \wedge \neg q)$ is

1	2	3	4	5	6	
p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg P \wedge \neg q$	$3 \wedge 6$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

The entries in the last column are 'F'. Hence $(p \vee q) \wedge (\neg p \wedge \neg q)$ is contradiction.

iii) The truth table is as follows.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \to q$	$(\neg p \wedge \neg q) \to (p \to q)$
T	T	F	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

⍰ All entries in last column are 'T'.

$\therefore$ $(\neg p \wedge \neg q) \to (p \to q)$ is a tautology.

iv) The truth table is as follows.

p	q	$\neg q$	$p \wedge \neg q$	$p \to q$	$(p \to q) \wedge (p \wedge \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

All the entries in the last column are 'F'. Hence it is contradiction.

v) The truth table for $\lceil\lfloor p \wedge (p \to \neg q) \to q \rceil\rfloor$

p	q	$\neg q$	$p \to \neg q$	$p \wedge (p \to \neg q)$	$\lceil\lfloor p \wedge (p \to \neg q) \to q \rceil\rfloor$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

The last entries are neither all 'T' nor all 'F'.

$\therefore$ $\lceil\lfloor p \wedge (p \to \neg q) \to q \rceil\rfloor$ is a neither tautology nor contradiction. It is a contingency.

2.6.2 Normal Form of a well formed formula :

One of the main problem in logic is to determine whether the given statement is a tautology or a contradiction. One method to determine it is the method of truth tables. Other method is to reduce the statement form to, called normal form.

If P & Q are two propositional variables we get various well formed formula.

The number of distinct truth values for formulas in P and Q is 2^4. Thus there are only 16 distinct formulae & any formula in P & Q is equivalent to one of these formulas.

Here we give a method of reducing a given formula to an equivalent form called a 'normal form'. We use 'sum' for disjunction, 'product' for conjunction and 'literal' either for P or for $\neg P$, where P is any propositional variable.

Elementary Sum & Elementary Product :

An elementary sum is a sum of literals. An elementary product is a product of literals.

e.g. $P \vee \neg Q$, $P \vee \neg R$ are elementary sum $P \wedge \neg Q$, $\neg P \wedge Q$ are elementary products.

Disjunctive Normal Form (DNF) :

A formula is in disjunctive normal form if it is a sum of elementary products.

e.g. $P \vee (\neg Q \wedge R)$, $P \vee (Q \wedge R)$

A conjunction of statement variables and their negations are called as fundamental conjunctions. It is also called min term.

e.g. P, $\neg P$, $P \wedge \neg Q$

Construction to obtain a Disjunctive Normal Form of a given formula

The following procedure is used to obtain a disjunctive normal form.

1. Eliminate $\rightarrow$ and $\leftrightarrow$ using logical identifies.
2. Use De-Morgans laws to eliminate $\neg$ before sums or products. The resulting formula has $\neg$ only before propositional variables i.e. it involves sum, product and literals.
3. Apply distributive laws repeatedly to eliminate product of sums. The resulting formula will be sum of products of literals i.e. sum of elementary products.

Example 16 :

Obtain a disjunctive normal form of

1. $P \wedge (P \rightarrow Q)$
2. $(P \rightarrow Q) \wedge (\neg P \wedge Q)$
3. $(P \wedge \neg(Q \wedge R)) \vee (P \rightarrow Q)$

Answer :

1) Consider, $P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$ (Implication law)

$(P \wedge \neg P) \vee (P \wedge Q)$ (Distributive law)

This is a disjunctive normal form of the given formula.

2) Using Implication law $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

$(P \rightarrow Q) \wedge (\neg P \wedge Q)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \wedge Q)$ Implication law

$\Leftrightarrow (\neg P \wedge Q) \wedge (\neg P \vee Q)$ Commutative law

$\Leftrightarrow (\neg P \wedge Q \wedge \neg P) \vee (\neg P \wedge Q \wedge Q)$ Distributive law

$\Leftrightarrow (\neg P \wedge \neg P \wedge Q) \vee (\neg P \wedge Q \wedge Q)$ Associative law

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge Q)$

This is required disjunctive normal form

3) $\left(p \wedge \neg(Q \wedge R)\right) \vee (P \to Q)$

$\Leftrightarrow \left(P \wedge \neg(Q \wedge R)\right) \vee (\neg P \vee Q)$ Implication law

$\Leftrightarrow \left(P \wedge (\neg Q \vee \neg R)\right) \vee (\neg P \vee Q)$ De-Morgans law

$\Leftrightarrow \left((P \wedge \neg Q) \vee (P \wedge \neg R)\right) \vee (\neg P \vee Q)$ Distributive law

$\Leftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R) \vee (\neg P \vee Q)$ Associative law

This is the disjunctive normal form of the given formula.

- Note that for the same formula we may get different disjunctive normal forms. So we introduce one or more normal forms called the principle disjunctive normal form or sum of products canonical form in the next definition. The advantage of constructing principle disjunctive normal form is that for a given formula principle disjunctive normal form is unique.

- Two forms are said to be equivalent iff their principle disjunctive normal forms consider.

*** Min term :**

A min term in n propositional variables P_1, P_2, ……, P_n is $Q_1 \wedge Q_2 \wedge .... \wedge Q_n$ where each Q_i is either P_i or $\neg P_i$.

e.g.

The min terms in P_1 & P_2 are $P_1 \wedge P_2$, $P_1 \wedge \neg P_2$, $\neg P_1 \wedge P_2$, $\neg P_1 \wedge \neg P_2$,

In general the number of min terms in n propositional variables is 2^n.

2.6.3 Principle Disjunctive Normal Form :

A formula a is in principle disjunctive normal form if a is a sum of min terms.

Steps to Construct Principle Disjunctive Normal Form of a given Formula : -

1. First obtain the disjunctive normal form for given formula.
2. Drop elementary products, which are contradiction such as $(P \wedge \neg P)$
3. If P_i & $\neg P_i$ are not present in an elementary product a, replace a by $(\alpha \wedge P_i) \vee (\alpha \wedge \neg P_i)$

4. Use the above step until all elementary products are reduced to sum of min terms.

   Use idempotent laws to avoid repetition of min terms.

Example 17 :

Obtain the canonical sum of product form i.e. principle disjunctive normal form of

1. $a \equiv P \vee (\neg P \wedge \neg Q \wedge R)$
2. a whose truth table is given below

Row No.	P	Q	R	a
1	T	T	T	T
2	T	T	F	F
3	T	F	T	F
4	T	F	F	T
5	F	T	T	T
6	F	T	F	F
7	F	F	T	F
8	F	F	F	T

Answer :

1) a is already in disjunctive normal form. There are no contradictions. So we have to introduce missing - variables.

$\neg P \wedge \neg Q \wedge R$ in a is a min - term.

As $P \equiv (P \wedge Q) \vee (P \wedge \neg Q)$

$$\therefore P \equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$$

Therefore the canonical sum of products form of a is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$$

2) For given a , we have T in column corresponding to rows 1, 4, 5 and 8. The min terms corresponding to these rows are $P \wedge Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge R$ and $\neg P \wedge \neg Q \wedge \neg R$

$\therefore$ The principle disjunctive normal form of a is

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

Fundamental disjunction (Max term)

A disjunction of statement variables and (or) their negations are called as fundamental disjunctions. It is also called **max term**.

e.g. $P, \neg P, \neg P \wedge Q, P \wedge Q, P \vee \neg P \vee Q$

Conjunctive Normal Form : -

A statement form which consists of a conjunction of a fundamental disjunction is called a conjunctive normal form.

e.g. $P \wedge Q, (P \vee Q) \wedge \neg P$

If a is in disjunctive normal form then $\neg a$ is in conjunctive normal form.

Maxterm

A max term in n propositional variables $P_1, P_2 \ldots\ldots P_n$ is $Q_1 \vee Q_2 \vee \ldots\ldots \vee Q_n$ where each Q_i is either P_i or $\neg p_i$

2.6.4 Principal Conjunctive Normal form :

A formula a is in principle conjugate normal form if a is a product of max terms. For obtaining the principle conjunctive normal form of a we can construct the principle disjunctive normal form of $\neg a$ and apply negation.

Example 18

Obtain a conjunctive normal form of

1. $a = P \vee (Q \rightarrow R)$
2. $a = (\neg P \rightarrow R) \wedge (P \leftrightarrow Q)$

1) Consider

$a = P \vee (Q \rightarrow R)$

$\neg a = \neg(P \vee (Q \rightarrow R))$

$\equiv \neg(P \vee (\neg Q \vee R))$ Implication law

$\equiv \neg P \wedge (\neg(\neg Q \vee R))$ De-Morgans law

$\equiv \neg P \wedge (Q \wedge \neg R)$ De-Morgans law & Double negation

$\therefore \neg a \equiv \neg P \wedge (Q \wedge \neg R)$

Hence, this is the required conjunctive normal form.

The principal conjugate normal form of a is $\neg(\neg p \wedge (Q \wedge \neg R)) = P \vee \neg Q \vee R$

2) $a = (\neg P \rightarrow R) \wedge (P \leftrightarrow Q)$

Since, we know that

$P \rightarrow Q \equiv \neg P \vee Q$ Implication law

$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$ Implication law

$\therefore a \equiv (\neg P \rightarrow R) \wedge ((P \rightarrow Q) \wedge (Q \rightarrow P))$

$$\equiv(\neg(\neg P)\vee R)\wedge((\neg P\vee Q)\wedge(\neg Q\vee P))$$
$$\equiv(P\vee R)\wedge((\neg P\vee Q)\wedge(\neg Q\vee P))$$
$$\therefore\ \equiv(P\vee R)\wedge(\neg P\vee Q)\wedge(\neg Q\vee P)$$

Which is required conjunctive normal form.

2.7 PREDICATES AND QUANTIFIERS

2.7.1 Predicates : A predicate is a statement containing one or more variables.

Proposition :

If values are assigned to all the variables in a predicate, the resulting statement is a proposition.

e.g.

1. $x<9$ is a predicate
2. $4 < 9$ is a proposition

Propositional Function :

Let a be a given set. A propositional function (or : on open sentence or condition) defined on A is an expression P(*x*) which has the property that P(a) is true or false for each $a \in A$.

The set A is called domain of P(*x*) and the set T_p of all elements of A for which P (a) is true is called the truth set of P(*x*).

i.e. $T_p=\{x : x \in A, p(x) \text{ is true}\}$ or $T_p = \{x : p(x)\}$ Another use of predicates is in programming Two common constructions are "if P(x), then execute certain steps" and "while Q(x), do specified actions." The predicates P(x) and Q(x) are called the guards for the block of programming code often the guard for a block is a conjunction or disjunction.

e.g. Let A = {*x* / *x* is an integer < 8}
Here P(*x*) is the sentence "*x* is an integer less than 8".

The common property is "an integer less than 8".
∴ P(1) is the statement "1 is an integer less than 8".
∴ P(1) is true, ∴ $1 \in A$ etc.

2.7.2 Quantifiers :

The expressions ' for all' and 'there exists' are called quantifiers. The process of applying quantifier to a variable is called quantification of variables.

Universal quantification :

The universal quantification of a predicate P(*x*) is the statement, "For all values of *x*, P(*x*) is true."

The universal quantification of P(*x*) is denoted by $\forall$ for all x P(*x*).

The symbol $\forall$ is called the universal quantifier.

e.g.

1) The sentence P(x) : - (-*x*) = *x* is a predicate that makes sense for real numbers *x*.

The universal quantification of P(*x*), $\forall$ *x* P(*x*) is a true statement because for all real numbers, -(- *x*) = *x*.

2) Let Q(*x*) : x + 1 < 5, then $\forall$ Q(*x*) : x + < 5 is a false statement, as Q(5) is not true. Universal quantification can also be stated in English as "for every *x*", "every *x*", or "for any *x*."

Existential quantification -

The existential quantification of a predicate P(*x*) is the statement "There exists a value of *x* for which P(*x*) is true."

The existential quantification of P(*x*) is denoted $\exists x P(x)$. The symbol $\exists$ is called the existential quantifier.

e.g.

1) Let $Q : x + 1 < 4$. The existential quantification of Q(*x*), $\exists x Q(x)$ is a true statement, because Q(2) is true statement.

2) The statement $\exists y$, y + 2 = y is false. There is no value of y for which the propositional function y+2=y produces a true statement.

Negation of Quantified statement :

$\neg(\forall\, x \in a)\, p(x) \equiv (\exists x \in A)\, \neg\, p(x)$

or $\neg\, \forall\, x\, p(x) \equiv \exists\, x\, \neg\, p(x)$

This is true for any proposition p(*x*). DeMorgan's Law.

2.7.3 The result for universal and existential quantifiers is as follows.

I) $\neg(\forall x \in A)\ p(x) \equiv (\exists x \in A) \neg p(x)$

In other words, the following two statements are equivalent.

i) It is not true that, for all $a \in A$, P(a) is true.

ii) There exists an $a \in A$, such that P(a) is false.

II) $\neg(\exists x \in A)\, p(x) \equiv (\forall x \in A)\neg p(x)$

That is, the following two statements are equivalent.

i) It is not true that for some $a \in A$, P(a) is true.
ii) For all $a \in A$, P(a) is false.

Other several properties for the universal and existential quantifiers are………

III) $\exists x(p(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x q(x)$

IV) $\exists x(P(x) \wedge Q(x)) \rightarrow \exists x P(x) \wedge \exists x Q(x)$ is a tautology.

V) $((\forall x p(x)) \vee (\forall x Q(x)) \rightarrow \forall x(p(x) \vee Q(x))$ is a tautology.

VI) $\forall x(P(x) \wedge Q(x) \equiv \forall x P(x) \wedge \forall x Q(x)$

VII) $\exists x(P(x) \vee Q(x) \equiv \exists x(P(x) \vee \exists x Q(x)$

Example 19 :

Express the statement using quantifiers. "Every student in your school has a computer or has a friend who has a computer."

Solution :

Let c(x) : "x has a computer"
F(x,y) : "x and y are friends"

$\therefore$ We have
$\forall x(c(x) \vee \exists y(c(y) \wedge F(x, y))$

Example 20 :

Express following using quantifiers.
i) There exists a polar bear whose colour is not white.
ii) Every polar bear that is found in cold region has a white colour.

Solution :

Let A(x) : x has a white colour
B(x) : x is a polar bear.
C(x) : x is found in cold region.
Over the universe of animals.

i) There exists a polar bear whose colour is not white.

$$\exists x(B(x) \wedge \neg A(x))$$

ii) Every polar bear that is found in cold regions has a white colour.

$$\forall x((B(x) \wedge c(x)) \rightarrow A(x)).$$

2.8 THEORY OF INFERENCE FOR THE PREDICATE CALCULAS

If an implication $P \rightarrow Q$ is a tautology where P and Q may be compound statement involving any number of propositional variables we say that Q logically follows from P. Suppose $P(P_1, P_2 \ldots\ldots P_n) \rightarrow Q$. Then this implication is true regardless of the truth values of any of its components. In this case, we say that q logically follows from P_1, P_2…..,P_n.

Proofs in mathematics are valid arguments that establish the truth of mathematical statements.

To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements. The rules of inference for statements involving existential and universal quantifiers play an important role in proofs in Computer Science and Mathematics, although they are often used without being explicitly mentioned.

2.8.1 Valid Argument :

An argument in propositional logic is a sequence of propositions.

All but the final propositions in the argument are called hypothesis or Premises.

The final proposition is called the conclusion.

An argument form in propositional logic is a sequence of compound propositions - involving propositional variables.

An argument form is valid if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

2.8.2 Rules of Inference for Propositional logic

We can always use a truth table to show that an argument form is valid. Arguments based on tautologies represent universally correct method of reasoning. Their validity depends only on the form of statements involved and not on the truth values of the variables they contain such arguments are called <u>rules of inference</u>.

These rules of inference can be used as building blocks to construct more complicated valid argument forms

e.g.

Let P: "You have a current password"

Q: "You can log onto the network".

Then, the argument involving the propositions,

"If you have a current password, then you can log onto the network".

"You have a current password" therefore: You can log onto the network" has the form …

.

$$\begin{array}{l} P \to Q \\ P \\ \hline \therefore Q \end{array}$$

Where $\therefore$ is the symbol that denotes 'therefore we know that when P & Q are proposition variables, the statement $((P \to Q) \land P) \to Q$ is a tautology.

$\therefore$ This is valid argument and hence is a rule of inference, called modus ponens or the law of detachment.

(Modus ponens is Latin for mode that affirms)

The most important rules of inference for propositional logic are as follows…..

	Rule of Inference	Tautology	Name
1)	$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$	$(P \land (P \to Q)) \to Q$	Modus ponens
2)	$\begin{array}{l} \neg Q \\ P \to Q \\ \hline \therefore \neg P \end{array}$	$[\neg Q \land (P \to Q)] \to \neg P$	Modus tollens
3)	$\begin{array}{l} P \to Q \\ Q \to R \\ \hline \therefore P \to R \end{array}$	$[(P \to Q) \land (Q \to R)] \to (P$	Hypothetical syllogism
4)	$\begin{array}{l} P \lor Q \\ \neg P \\ \hline \therefore Q \end{array}$	$[(P \lor Q) \land \neg P] \to Q$	Disjunctive syllogism

5)	P $\therefore PVQ$	$P \rightarrow (P \vee Q)$	Addition
6)	$P \wedge Q$ $\therefore P$	$(P \wedge Q) \rightarrow P$	Simplification
7)	P Q $\therefore P \wedge Q$	$((P) \wedge (Q)) \rightarrow P \wedge Q$	Conjunction
8)	$P \vee Q$ $\neg P \vee R$ $\therefore Q \vee R$	$[(P \vee Q) \wedge (\neg P \vee R)] \rightarrow (Q \vee R$	Resolution

Example 21 :

Show that $R \rightarrow S$ can be derived from the premises (i) $P \rightarrow (Q \rightarrow S)$ (ii) $\neg(R \vee P)$ and iii) Q.

Solution :

The following steps can be used to establish the conclusion.

	Steps	Reason
1	$P \rightarrow (Q \rightarrow S)$	Premise (i)
2	$R \vee P$	Premise (ii)
3	$R \rightarrow P$	Line 2, implication
4	$R \rightarrow (Q \rightarrow S)$	Hypothetical Syllogism
5	$R \rightarrow (\neg Q \vee S)$	Line 4, implication
6	$\neg R \vee (\neg Q \vee S)$	Line 5, implication
7	Q	Premise (iii)
8	$\neg R \vee S$	Line 6, 7 and Disjunctive syllogism
9	$R \rightarrow S$	Line 8, implication

Hence the proof :

Example 22 :

Test the validity of the following arguments :
1. If milk is black then every crow is white.
2. If every crow is white then it has 4 legs.
3. If every crow has 4 legs then every Buffalo is white and brisk.
4. The milk is black.
5. So, every Buffalo is white.

Solution :

Let P : The milk is black
Q : Every crow is white
R : Every crow has four legs.
S : Every Buffalo is white
T : Every Buffalo is brisk

The given premises are

(i) $P \to Q$

(ii) $Q \to R$

(iii) $R \to S \wedge T$

(iv) P

The conclusion is S. The following steps checks the validity of argument.

1.	$P \to Q$	… premise (1)
2.	$Q \to R$	… Premise (2)
3.	$P \to R$	… line 1. and 2. Hypothetical syllogism (H.S.)
4.	$R \to S \wedge T$	… Premise (iii)
5.	$P \to S \wedge T$	… Line 3. and 4.. H.S.
6.	P	… Premise (iv)
7.	$S \wedge T$	Line 5, 6 modus ponets
8.	S	Line 7, simplification
$\therefore$	The argument is valid	

Example 23 :

Consider the following argument and determine whether it is valid or not. Either I will get good marks or I will not graduate. If I did not gradute I will go to USA. I get good marks. Thus, I would not go to USA.

Solution :

Let P : I will get good marks.
Q : I will graduate.
R : I will go to USA

The given premises are

i) $P \vee \neg Q$

ii) $\neg Q \to R$

iii) P

The conclusion is $\neg R$.
The following steps checks is validity.

	Steps	Reason
1.	$P \vee \neg Q$	… premise (i)
2.	$\neg\neg P \vee \neg Q$	…Double negation
3.	$\neg P \to \neg Q$	Line 2, Implication
4.	$\neg Q \to R$	… premise (ii)

5.	$\neg P \rightarrow R$	Line 3, 4, H.S.
6.	P	Premise (iii)
7.	R	Line 5 implication and line 6
8.	Conclusion is R or $\neg$ R	Line 7 simplification

$\therefore$ The argument is not valid

2.9 MATHEMATICAL INDUCTION

Here we discuss another proof technique. Suppose the statement to be proved can be put in the from $\forall$ $n \geq n_0$. P(n), where n_0 is some fixed integer.

That is suppose we wish to show that P(n) is true for all integers $n \geq n_0$.

The following result shows how this can be done.

Suppose that

(a) $P(n_0)$ is true and

(b) If P(K) is true for some $K \geq n_0$, then P(K + 1) must also be true. The P(n) is true for all $n \geq n_0$.

This result is called the principle of Mathematical induction.

Thus to prove the truth of statement $\forall$ $n \geq n_0$. P(n), using the principle of mathematical induction, we must begin by proving directly that the first proposition $P(n_0)$ is true. This is called the basis step of the induction and is generally very easy.

Then we must prove that $P(K) \Rightarrow P(K + 1)$ is a tautology for any choice of $K \geq n_0$. Since, the only case where an implication is false is if the antecedent is true and the consequent is false; this step is usually done by showing that if P(K) were true, then P(K + 1) would also have to be true. This step is called induction step.

In short we solve by following steps.

1. Show that P(1) is true.
2. Assume P(k) is true.
3. Prove that P(k +1) is true using P(k)

Hence P(n) is true for every n.

Example 24 :

Using principle of mathematical induction prove that…

1) $1 + 2 + 3 + \ldots.. + n = \frac{n(n+1)}{2}$ for all $n \geq 1$

2) $n^3 - n$ is divisible by 3 for $n \in Z^+$

3) $2^n > n$ for all positive integers n.

4) $n! \geq 2^{n-1}$

5) If $A_1, A_2, \ldots\ldots A_n$ be any n sets then $\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}$

Solution :

For all n, 1) Let $P(n): 1 + 2 + 3 + - - - + n = \frac{n(n+1)}{2}, n \geq 1$

<u>Step 1</u> : Here $n_0 = 1$

We must show that P (1) is true.

P (1) is the statement

$1 = \frac{1(1+1)}{2}$

Which is clearly true.

Hence P(1) is true.

<u>Step 2</u> :

Assume P(K) is true for K ≤ n.

$\therefore$ $P(K) \equiv 1 + 2 + \ldots.. + K = \frac{K(K+1)}{2}$ K ≥ 1 …..(1)

<u>Step 3</u> :

To show that P(K + 1) is true.

$\therefore$ $P(K+1) = 1 + 2 + \ldots.. + (K+1) = \frac{(K+1)((K+1)+1)}{2}$

Consider,

$1 + 2 + \ldots.. + (K + 1) = 1 + 2 + \ldots.. + K + (K+1)$

$= \frac{K(K+1)}{2} + (K+1)$ using eqn. (1)

$\therefore$ $1 + 2 + \ldots.. + (K+1) = \frac{K(K+1) + 2(K+1)}{2}$

$= \frac{(K+1) + (K+2)}{2}$

$= \frac{(K+1) + ((K+1)+1)}{2}$

Which is RHS of P(K + 1)

Thus, P(K + 1) is true.

$\therefore$ By principle of mathematical induction it follows that P(n) is true for all n≥1.

$\therefore$ $1 + 2 + \ldots.. + n = \frac{n(n+1)}{2}$

2) Let P(n) : $n^3 - n$ is divisible by 3.

Step 1 : We note that,
P(1) : $1^3 - 1 = 0$ is divisible by 3
$\therefore$ P(1) is true.

Step 2 :
Assume P(K) is true for $K \le n$
$\therefore$ P(K): $K^3 - K$ is divisible by 3.
We can write $K - k = 3m$ for $m \in N$. ……(1)

Step 3 :
We prove that P(K + 1) is true.
P(K + 1); $(K + 1)^3 - (K + 1)$ is divisible by 3.
Consider

$$\begin{aligned}(K + 1)^3 - (K + 1) &= K^3 + 3K^2 + 3K + 1 - K - 1 \\ &= K^3 + 3K^2 + 2K \\ &= 3m + K + 3K^2 + 2K \quad \text{(using (1))} \\ &= 3(m + K + K^2)\end{aligned}$$

Hence $(K + 1)^3 - (K + 1)$ is divisible by 3.

Thus, P(K + 1) is true when P(K) is true.

$\therefore$ By principle of mathematical induction the statement is true for every positive integer n.

3) Let P(n) : $2^n > n$ $\forall$ positive integer n.

Step I : For n = 1, $2^1 = 2 > 1$
Hence P(i) is true.

Step II : Assume P(K) is true for every positive integer K i.e.

$$2^K > K \qquad …..(1)$$

Step III : To show that P(K + 1) is true
From (1),

$$2^K > K$$

Multiplying both sides by 2, we get,

$$2.2^K > 2.K$$
$$\therefore \quad 2^{K+1} > 2K$$
$$\therefore \quad 2^{K+1} > K + K > K + 1$$

$\therefore$ P(K + 1) is true when P(K) is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.
$\therefore$ $2^n > n$ for positive integer n.

4) Let P(n) : $n! \ge 2^{n-1}$

Step I : For n = 1

$1! = 1 \geq 2^{1-1} = 2^0 = 1$

$\therefore$ P(1) is true.

Step II : Assume P(K) is true for some K < n.

$\therefore K! \geq 2^{k-1}$ …..(1)

Step III : Prove that P(K + 1) is true.

Consider $K! \geq 2^{k-1}$ (from (1))

As $K + 1 \geq 2$

$\therefore$ $K! \geq 2^{k-1}$ and $K + 1 \geq 2$

Taking the product we get,

$K! \times (K + 1) \geq 2^{k-1} \times 2$

$\therefore$ $(K + 1)K! \geq 2^{k-1+1}$

$\therefore$ $(K + 1)! \geq 2^k$

Hence P(K + 1) is true.

$\therefore$ By principle of mathematical induction P(n) is true for every n.

5) Let $P(n) : \overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}$

Step I : For n = 2,

LHS $= \overline{\left(\bigcup_{i=1}^{2} A_i\right)} = \overline{(A_1 \cup A_2)} = \overline{A_1} \cap \overline{A_2}$

& RHS $= \bigcap_{i=1}^{2} \overline{A_i} = \overline{A_1} \cap \overline{A_2}$

$\therefore$ LHS = RHS

Hence P(2) is true.

Step 2 : Assume P(K) is true for some K < n

$\therefore$ $\overline{\left(\bigcup_{i=1}^{k} A_i\right)} = \bigcap_{i=1}^{k} \overline{A_i}$ …..(1)

Step 3 : Prove that P(K + 1) is true.

Consider

$$\overline{\left(\bigcup_{i=1}^{k+1} A_i\right)} = \overline{\left(\bigcup_{i-1}^{k} A_i \cup A_{k+1}\right)} = \overline{\left(\bigcup_{i-1}^{k} A_i\right)} \cap \overline{A_{k+1}}$$

$$= \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}} \quad \text{(from (1))}$$

$$= \bigcap_{i=1}^{k+1} \overline{A_i}$$

$\therefore$ P(K + 1) is true

$\therefore$ By principle of mathematical induction P(n) is true for all n.

$$\therefore \quad \overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \bar{A}_i$$

2.10 UNIT AND EXERCISE :

1. Construct the truth table of

   $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$ [Jan. 11]

2. Construct the truth table of

   $(Q \wedge P) \vee (Q \wedge \neg P)$ [Dec. 09]

3. Construct the truth table for each of the following.

   i) $(P \rightarrow Q) \vee (\neg P \rightarrow Q)$

   ii) $P \leftrightarrow \neg P$

   iii) $(P \vee Q) \wedge \neg R$

   iv) $P \rightarrow (\neg Q \vee R)$

   v) $(PQ) \wedge (\neg P \rightarrow R)$

4. Show that P V $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent.

5. Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \equiv R$ [Jan. 11]

6. Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

7. Determine whether $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is a tautology or contradiction or neither. [May 10]

8. Obtain the conjunctive normal form of

   $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$ [Jan. 2011]

9. Obtain conjunctive and disjunctive normal form of the following.

   i) $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$ [May 10]

   ii) $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow Q)$ [Dec. 09]

   iii) $P \vee (Q \rightarrow R)$

   iv) $\neg (P \vee Q) \leftrightarrow (P \wedge Q)$

   v) $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

10. Objtain principle disjunctive and conjunctive normal form of

    i) $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

    ii) $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow Q)$

11. Obtain a conjunctive normal form of $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ show that it is a tautology.
12. What is quantifier ? Explain with suitable examples.
13. Check the validity of following argument "If Anand has completed M.C.A. or M.B.A. Then he is assured a good job. If Anand is assured a good job, he is happy. Anand is not happy. So Anand has not completed M.C.A."
14. Show that conclusion S follows from the premises $(P \rightarrow Q) \wedge (P \rightarrow R)$, $\neg (Q \wedge R)$ and S V P.
15. Express the following using quantifiers.
    - i) Every student in the college has a computer or has a friend who has a computer.
    - ii) All rational numbers are real numbers.
    - iii) Some rational numbers are not real.
    - iv) All men are mortal.
    - v) Some women are beautiful.
16. Using Principle of mathematical induction prove that $n^3 + 2n$ is divisible by 3 for every positive integer n.
17. Prove by mathematical induction that $2^n < n!$ for $n \geq 4$.
18. Show by mathematical induction that for all $n \geq 1 1^2 + 2^2 + 3^2 + - - - + n^2 = \frac{n(n+) (2n+1)}{6}$
19. Prove by mathematical induction that 3 / $(n^3 - n)$ for every positive inter n.
20. Prove by mathematical induction
    - i) $5^n + 3$ is divisible by 4.
    - ii) $n^2 + n$ is always even.
    - iii) Let P(n) : $1^3 + 2^3 + 3^3 + \ldots .. + n^3 = \frac{n^2 (n+1)^2 + 4}{4}$

      a) Use P(k) to show P(k+1)

      b) Is P(n) true for all $n \geq 1$

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